Section 3-4, Mathematics 108

Real Zeros of Polynomials

Let's say you have a 3rd degree polynomial with real roots 2, 3, and -4.

So we have $P(x) = (x-2)(x-3)(x+4) = x^3 - x^2 - 14x + 24$

Note that 2, 3, and -4 are factors of 24.

Also note that 2+3+-4=-1 which happens to be the coefficient of x^2.
Could this be a coincidence?

Consider this polynomial where the *a*'s and *b*'s are integers.

$$(a_0 x + b_0)(a_1 x + b_1)\ldots(a_n x + b_n)$$

If we were to multiply this out the leading term would be

$$a_0 a_1 \ldots a_n x^n$$

and the constant term would be

$$b_0 b_1 \ldots b_n$$

Maybe that is not completely obvious, so let's try a few examples:

$$(a_0 x + b_0)(a_1 x + b_1) = a_0 a_1 x^2 + (a_0 b_1 + a_1 b_0)x + b_0 b_1$$

So this is true for a quadratic.

Let's take one more look at a 3rd degree polynomial

$$\left[a_0 a_1 x^2 + (a_0 b_1 + a_1 b_0)x + b_0 b_1\right](a_2 x + b_2)$$

after much effort this becomes

$$a_0 a_1 a_2 x^3 + (a_0 a_1 b_2 + a_0 a_2 b_1 + a_1 a_2 b_0)x^2 + (a_0 b_1 b_2 + a_1 b_0 b_2 + a_2 b_0 b_1)x + b_0 b_1 b_2$$

Note that if all the *a*'s are 1, then as mentioned, the coefficient of the send term is the sum of the negative of the sum of the roots.

The Rational Root Theorem

Note in the above polynomials the roots are

$$\frac{-b_0}{a_0}, \frac{-b_1}{a_1}, \ldots \frac{-b_n}{a_n},$$

Compare this with the ratio of the constant term divided by the leading coefficient.

$$\frac{b_0 b_1 \ldots b_2}{a_0 a_1 \ldots a_n}$$

Each of the roots is a factor of the constant term divided by the leading coefficient.

The Rational Root Theorem uses this fact to give us a way to look for any roots that happen to be rational.

Example:

Find the roots of $P(x) = x^3 - 3x + 2$.

The rational root theorem tells us that the possible rational roots are

$$\pm\frac{1}{1}\, and \pm\frac{2}{1}$$

Plugging each of them in we find

$$P(1) = 0$$
$$P(-1) = 4$$
$$P(2) = 4$$
$$P(-2) = 0$$

So 0, and -2 are the rational roots.

If we divide the polynomial by x-1 and x+2 we find the quotient is x-1 meaning that 1 is a duplicate root.

Example:

$$P(x) = 2x^3 + x^2 - 13x + 6$$

The possible rational roots are

$$\pm\frac{1}{1}, \pm\frac{2}{1}, \pm\frac{3}{1}, \pm\frac{6}{1}, \pm\frac{1}{2}, \pm\frac{2}{2}, \pm\frac{3}{2} \; and \; \pm\frac{6}{2}$$

This list has duplicates so lets simplify it

$$\pm 1, \pm 2, \pm 3, \pm 6, \pm\frac{1}{2} \; and \; \pm\frac{3}{2}$$

Let's try a few possibilities

$$P(1) = -4$$
$$P(2) = 0$$

Since we now have a root, we can divide by x-2

$$\frac{2x^3 + x^2 - 13x + 6}{x - 2} = 2x^2 + 5x - 3$$

At this point we can use the quadratic formula to find the remaining two roots

$$x = \frac{-5 \pm \sqrt{25 - 24}}{4} = \frac{-5 \pm 1}{4} = -1, -\frac{3}{2}$$

So the roots are 2, -1, and $-\frac{3}{2}$

Other Techniques described by the Book

Having a list of possible rational roots, the book describes two techniques that can narrow the list down.

Descarte's Rule of Signs

and the

Upper and Lower Bounds Theorem

The former gives some guidance as to how many of the roots will be negative and how many will be positive.

The latter will provide upper and lower limits on the roots.

You are welcome to review these two tools, however I will not assign them nor expect anyone to learn them.   The reason is simple.   They are quite complex to learn, and they don't allow you to solve any problems solvable with the rational root theorem, along with polynomial division.

The book does however suggest that using a computer generated graphing tool may be helpful.

Example:

What are the roots of

$$3x^4 + 4x^3 - 7x^2 - 2x - 3 = 0$$

Try using Desmos   finding 1.302779, and -2.30277